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Nonlinear flows in nearly incompressible hydrodynamic fluids

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Nearly incompressible viscous hydrodynamic fluids are investigated using nonlinear fluid simulations. Nearly incompressible fluids possess acoustic modes through high frequency fluctuations associated with the subsonic fluid Mach number. These modes, in combination with the fluid modes, drive linearly unstable modes and nonlinearly excite flows. The nonlinear flows damp the long wavelengths in our simulations, and are dissipated resonantly when certain nonlinear conditions are satisfied. In agreement with our analytic analysis, the nonlinearly saturated flows in nearly incompressible fluids are generated through the action of the Reynolds stress forces.

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I. INTRODUCTION

One of the most remarkable of astrophysical observations is the Kolmogorov-like interstellar (electron) density spectrum [1], extending over decades and decades in wave number space. Remarkably, similar density spectra are observed in the solar wind [2] as well. The ubiquity of the Kolmogorov-like density spectrum led Montgomery *et al.* [3] to suggest an explanation based on coupling incompressible MHD fluctuations to density fluctuations through a "pseudosound" relation [4]. The precise nature of the relationship between an incompressible fluid description and compressible fluctuations was elucidated with the development of "nearly incompressible" (NI) hydrodynamics and MHD [2,5–10]. Through the development of a careful expansion technique, the low (turbulent) Mach number fluid equations can be expanded to include the effects of acoustic fluctuations as leading-order corrections to the incompressible fluid model. The resulting equations [9] comprise the familiar incompressible hydrodynamical and MHD equations at leading order, together with a modified set of compressible hydrodynamical equations in which sources due to background incompressible fluid modes drive linearly unstable modes. Thus, the NI fluid equations retain compressibility to first order, producing (magneto)acoustic modes as well as convective modes. NI hydro and MHD have been surprisingly successful in the solar wind where predicted correlation are seen frequently [5,8] and predicted anisotropies are observed [11]. However, the basic nonlinear development of NI hydro and MHD remain completely unexplored and this report presents the first fully self-consistent investigation of NI hydrodynamics. We find that NI hydrodynamics admits remarkably rich and complex phenomena.

In this paper we explore the nearly incompressible hydrodynamic equations using nonlinear fluid simulations. We focus primarily on wave-wave interactions between nearly incompressible fluctuations driven by background viscous incompressible fluctuations. The NI fluid models incorporating thermal transport are considered elsewhere. Our simulations show that such interactions lead to nonlinearly generated flows through the action of the Reynolds stresses on the fluid. This discovery is supported by our analytical studies of the zero-frequency component of the nonlinearly saturated flows.

The rest of the paper is organized as follows. The basic equations of the NI model are described in Sec. II. The linear modes giving rise to acoustic waves in the low Mach number fluid and the instability excited nonlinearly by the $k_y=0$ mode are described in Sec. III. Section IV deals with nonlinear fluid simulations that demonstrate the excitation of nonlinearly saturated flow ($k_y=0$ mode) instability. A theoretical basis for understanding the nonlinear flow mode in our nonlinear fluid simulations is also discussed. The mode coupling calculation explains qualitatively the interaction between nonlinear flow mode and the underlying turbulence. Conclusion are discussed in Sec. V.

II. MODEL EQUATIONS

The set of NI hydrodynamics fluid equations derived by Zank *et al.* [9] couple convective fluid motion with high frequency acoustic fluctuations describing appropriately the high β (where β is the ratio of plasma and magnetic field pressures) interstellar plasmas. The background incompressible fluid can be described by the usual equations of incompressible hydrodynamics,

$$\frac{\partial}{\partial t}\mathbf{U}^\infty + \mathbf{U}^\infty \cdot \nabla \mathbf{U}^\infty = -\nabla p^\infty + \mu \nabla^2 \mathbf{U}^\infty, \quad \nabla \cdot \mathbf{U}^\infty = 0. \tag{1}$$

Here, the superscript ∞ indicates that the velocity $\mathbf{U}^\infty$ and the pressure p^∞ variables satisfy the incompressible fluid equations Eq. (1). The incompressible pressure satisfies $\nabla^2 p^\infty = -\nabla \cdot (\mathbf{U}^\infty \cdot \nabla \mathbf{U}^\infty)$. The weakly perturbed compressive fluctuations about the incompressible modes (denoted by superscript ∞) for velocity, pressure, and density variables are represented by $\mathbf{U}=\mathbf{U}^\infty+\epsilon\mathbf{U}_1$, $p=1+\epsilon^2(p^\infty+p^*)$, and $\rho=1+\epsilon^2\rho_1$, respectively. The nonlinear fluid equations describing the dynamical evolution of the compressible fluctuations in the NI hydrodynamical description [6,9] contain the compressible

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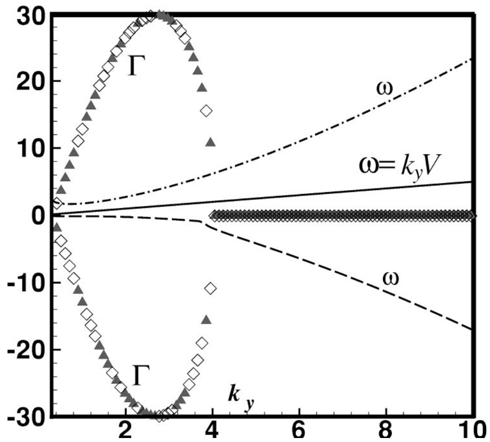

FIG. 1. Linear frequency spectrum describing normal and unstable modes of NI hydrodynamics. Acoustic modes in the NI fluids are represented by dashed-dotted, and long-dashed curves and the linear frequency of the incompressible fluid modes (solid line). The linear growth rate of NI modes are represented by hollow squares and filled triangles (on the right y-axis between +1 to −1).

fluid velocity $\mathbf{U}_1$ the compressible pressure p^* and the density ρ_1, which satisfy

$$\frac{\partial}{\partial t}\mathbf{U}_1 + \mathbf{U}^\infty \cdot \nabla \mathbf{U}_1 + \mathbf{U}_1 \cdot \nabla \mathbf{U}^\infty = - \nabla p^* + \nu \nabla^2 \mathbf{U}^\infty, \quad (2)$$

$$\frac{\partial p^*}{\partial t} + \mathbf{U}^\infty \cdot \nabla p^* + \nabla \cdot \mathbf{U}_1 = - \frac{\partial p^\infty}{\partial t} - \mathbf{U}^\infty \cdot \nabla p^\infty, \quad (3)$$

$$\frac{\partial \rho_1}{\partial t} + \mathbf{U}^\infty \cdot \nabla \rho_1 + \nabla \cdot \mathbf{U}_1 = 0. \quad (4)$$

The above equations are normalized, and correspond to their respective unnormalized variables as follows; $\mathbf{U}_1/M_s = \overline{\mathbf{U}}_1, \gamma^{1/2}p^*/\rho_0 = \overline{p^*}, \gamma^{1/2}\rho_1/\rho_0 = \overline{\rho_1}, \nu/\gamma^{1/2}M_s^2 = \overline{\nu}.$ The time and space coordinates are normalized by characteristic time and length scales, respectively $L\nabla = \overline{\nabla}, u_0 t/L = \overline{t}.$ Note that the bars have been removed from all the normalized variables for the sake of convenience. Here $M_s = u_0/C_s$ and $\epsilon^2 = \gamma M_s^2$, where M_s is the fluid Mach number, γ is the ratio of the specific heats, and C_s is the acoustic speed associated with the sound waves, $C_s^2 = \gamma p_0/\rho_0$.

III. LINEAR INSTABILITY

A linearized dispersion relation, about constant speed along the y direction, obtained from the system of Eqs. (1)–(4) clearly indicates the presence of the high frequency component of the acoustic modes in the nearly incompressible hydrodynamic fluid. These modes appear in the subsonic hydrodynamic fluid through a first order expansion of the fluid variables and represent compressible effects to the lowest order [9]. The dispersion relation [12], in general, is rather complex, and is shown in Fig. 1 for the simple case when $k_x=0$. The real frequency varies quadratically with k_y because of the coupling between incompressible and acoustic fluctuations (shown by the dashed-dotted, and long-dashed lines in Fig. 1) unlike the purely incompressible linear frequency mode (the solid line in the Fig. 1). The linear growth rate (curves with squares and triangles in Fig. 1) indicates that no linear instability can occur for the $k_y=0$ mode (essentially the nonlinear flow mode), although such a mode could possibly be generated via inverse cascading processes that are inherently governed by the nonlinear interactions. On the other hand, finite k_y modes can give rise to linear instabilities in the form of streamerlike structures. The linear growth rate for high k-modes is stabilized by viscous effects. The nearly incompressible equations are therefore a modified Eulerian system, which contains acousticlike waves, and linearly unstable modes, unlike the Euler fluid equations, Eq. (1). The dominant nonlinear interactions are due to convective nonlinear effects, Reynolds forces, such as $\mathbf{U}^\infty \cdot \nabla \mathbf{U}_1, \mathbf{U}_1 \cdot \nabla \mathbf{U}^\infty$, etc., all associated with the incompressible background fluid. The nonlinear interactions evidently couple the incompressible fluid modes with the acoustic component in a complex manner which is best revealed through nonlinear fluid simulations.

IV. NONLINEAR FLOWS

We have developed a nonlinear code (NIH) to solve the nonlinearly coupled set of NI and IN equations [i.e., Eqs. (1)–(4)] in two spatial dimensions [12]. The code is based upon a Fourier harmonic expansion of the evolution variables, using a 2/3 dealiased pseudospectral method in space. The time integration used a Runge-Kutta fourth order method, with 6×10^{-10} accuracy in the time step. Periodic boundary conditions are imposed along the x and the y directions. The initial states of both fluids, i.e., IN and NI, are set identically to coherent waves [Fig. 2(a)].

As the NI fluid is driven by IN-fluid modes, the NI-fluid vortices are, first, subject to the linear instability and begin to form streamerlike structures (radially extended) for which the k_y modes are finite. Hence the vortices appear to be shrunk in the y direction, and are elongated in the x direction.

With the saturation of the linear instabilities, nonlinear interactions begin to dominate the dynamics. Under the influence of nonlinear instabilities, the streamerlike vortices become unstable to nonlinear perturbations and further evolve through coalescence and merging processes to form nearly y-independent (i.e., $k_y\approx 0$) flows that vary along the x direction ($k_x\neq 0$) [see Fig. 2(c)]. Such flows are commonly known as nonlinear flows (radially localized, poloidally elongated, i.e., $k_y\approx 0$ modes) in the literature, and have attracted a great deal of attention in magnetically confined fusion plasmas [13]. The fluid density fluctuations propagate as compressional waves along the y direction through alternate compression and rarefaction processes (not shown here). The propagation of these waves in a preferential direction (i.e., along the y direction) is due to the nonlinearly generated flows which convect the density fluctuations parallel to them [see Eq. (4)]. The energy associated with the entire evolution is depicted in Fig. 3, where the normalized kinetic energies (KE) of the two fluids are equal at $t=0$. For very small but finite viscosities (μ & ν), the KE associated with the IN fluid

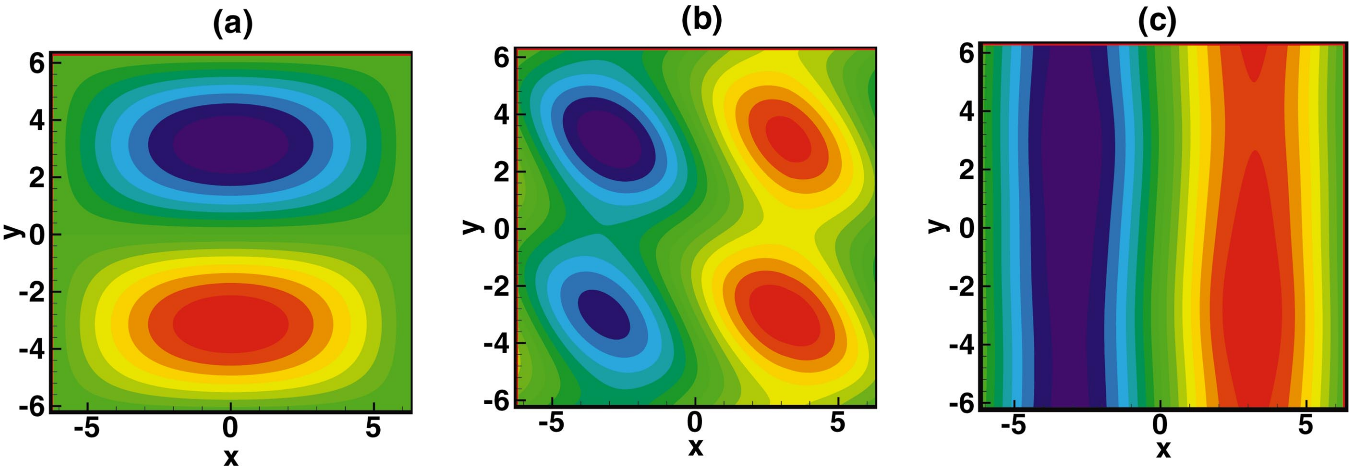

FIG. 2. (Color) Evolution of waves in NI fluid. Shown here are the constant contours of fluid velocity $|\mathbf{U}_1|$. (a) Initial states for both IN and NI fluid modes consist of large scale coherent waves. (b) End of linear phase shows formation of streamer-like vortices, which further merge with each other due to nonlinear interactions, and form nonlinear flows mode as in (c).

remains almost constant, while it grows linearly for the NI fluids due to unstable modes. It is observed in our simulation that the energy associated with the nonlinearly generated flow mode provides the dominant contribution to the KE of the NI fluid. The total KE of the NI fluid evolves through three different stages, which repeat themselves periodically. These correspond to nonlinear growth, saturation, and dissipation stages. During these stages the modes, driven by linear instabilities, acquire nonlinear amplitudes and eventually excite nonlinear flows. The flows thereafter quench the turbulence in the saturation stage, and consequently the energy associated with the turbulence falls off. The nonlinear flows are destabilized further by the nonlinear instability mechanism and are dissipated. At the same time, the turbulence grows and the entire dynamics repeats itself as described above. Thus the turbulence and the flows regulate each other quite systematically and their interplay is shown in the Fig. 4. We further find that this phenomena is generic to a collection of large number of interacting waves.

To understand the mechanism leading to the generation of nonlinearly excited flows observed in our simulations, consider the nonlinear flow modes of zero frequency and $k_y=0$. For this purpose, we first take the curl of Eqs. (1) and (2) to eliminate the pressures p^∞ and p^*, and obtain the incompressible and nearly incompressible vorticity equations by expressing the respective velocity fields in terms of scalar functions. The incompressible and nearly incompressible velocities are therefore represented as $\mathbf{U}^\infty=\hat{z}\times\nabla\phi$ and $\mathbf{U}_1=\hat{z}\times\nabla\psi+\nabla\psi$. This allows us to retain the effect of compressibility in the nearly incompressible hydrodynamic fluid. A reductive perturbation method [14] is then applied to the set of nonlinearly coupled NI and IN fluid vorticity equations. The underlying method involves expansion of the dependent variables in orders of ϵ and equating terms of equal order. The higher order terms in the expansion then yield the zero frequency component (i.e., $l=0$ mode), which essentially corresponds to the nonlinearly saturated amplitude of the nonlinear flows. Using this method, we calculate the saturated potential of zero frequency, i.e., the flow. The variables are expanded in spherical harmonics using $\Theta=\sum_\alpha \epsilon^\alpha\Theta^{(\alpha)}$, with $\Theta^{(\alpha)}=\sum_\ell \Theta_\ell^{(\alpha)}(x,\xi,\tau)\exp[i\ell(k_y y-\omega t)]$, where Θ corresponds to the incompressible and the nearly incompressible velocity potentials ϕ and ψ variables, ϵ is a

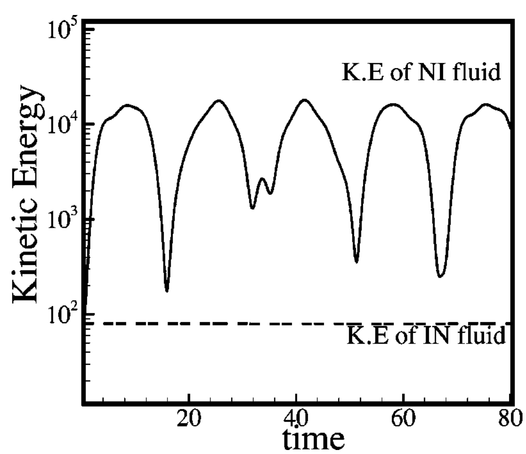

FIG. 3. The kinetic energy associated with the IN (dashed-line) and NI (solid line) fluids.

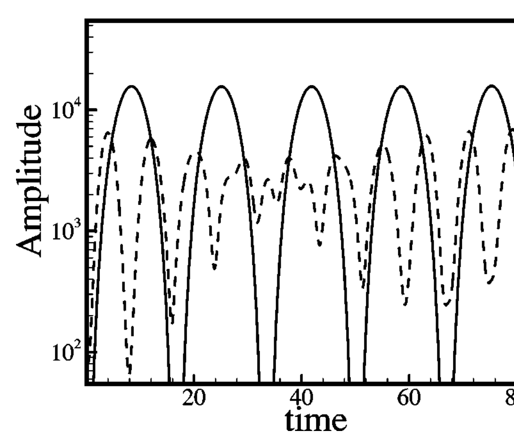

FIG. 4. The nonlinear flow mode (solid curve) and turbulence (dashed curve) are shown in an arbitrary units.

smallness parameter for the amplitude of the variables, $\xi = \epsilon(y-ut)$, and $\tau=\epsilon^2 t$. The two fluid viscosities μ and ν are ordered as $\sim(\epsilon^2)$. The amplitudes are subject to the reality condition, $\Theta_\ell^{(\alpha)}=\Theta_{-\ell}^{(\alpha)^*}$ and $\Theta_\ell^{(1)}=0$ for $\ell=\pm 1$. The first order ϵ^1 equation then readily yields $iD_\ell(\omega,k)\tilde{\psi}_\ell^{(1)}=0$ which is the dispersion relation for $\ell=1$, i.e., $D_1(k,\omega)=\omega-k_y v$. Here we have assumed a sinusoidal dependence for the perturbed variables, $\phi_\ell^{(1)}=\tilde{\phi}_\ell^{(1)}(\xi,\tau)\sin k_m x$, and $\psi_\ell^{(1)}=\tilde{\psi}_\ell^{(1)}(\xi,\tau)\sin k_m x$, where $k_m=m(2\pi/L)(m=1,2,\ldots)$ and L is the length of the system. To second order, the vorticity equations introduce terms of the type $\mathbf{U}^\infty\cdot\boldsymbol{\nabla}\mathbf{U}_1$, which survive only for complex ω. These are transport fluxes. In our treatment, we have retained the sources to ϵ^2 and higher orders. The transport fluxes due to nonlinear terms arise only at the ϵ^2 and higher orders. We, however, omit these terms since they are balanced by sources in the steady state. The following equation is then obtained, $iD_l(\omega,k)\tilde{\psi}_l^{(2)}+\partial D_l/\partial k_y \partial\tilde{\psi}_l^{(1)}/\partial\xi=0$.

At the third order, we obtain the zero-frequency component of the nearly incompressible fluid, i.e., the nonlinear flow mode ($l=0$ mode) as

$$\tilde{U}_{1_0}^{(2)} = 2\frac{k_y^2}{k_m^2}\frac{v}{u}\left(\frac{4k_y\omega_r}{\omega_r^2+\gamma^2}+\frac{1}{u-v}\right)\frac{\partial^2}{\partial x^2}|\tilde{\phi}_1^{(1)}|^2, \tag{5}$$

where u is the group velocity of the fluid. Here ω_r and γ are, respectively, the real frequency and the growth rate of the nearly incompressible fluctuations. In arriving at Eq. (5), we make use of the $\tilde{\phi}_0^{(2)}$ component from the incompressible fluid vorticity equation to evaluate the nonlinear terms of the nearly incompressible fluid vorticity, which primarily results from the Reynolds stress forces in the fluid equations. We therefore see, analytically, that the generation of flow in our fluid simulations is a consequence of the Reynolds stresses that are proportional to $\sim|\tilde{\phi}_1^{(1)}|^2$, and that the flows vary along the x direction. As observed in our simulation the nonlinear flow mode, in the saturated state, is destabilized. To understand what damps the flow mode, we derive the nonlinear mode coupling equations for the flow from the IN and NI fluid vorticity equations,

$$\begin{aligned}\frac{\partial \boldsymbol{\Psi}_k}{\partial t} - \nu k_x^4\tilde{\phi}_{\mathbf{k}}(t) = \sum_{\mathbf{k}=\mathbf{k_1}+\mathbf{k_2}} \frac{k_{2y}}{k_x}\{&k_{2x}^2\tilde{\phi}_{\mathbf{k_1}}(t)\tilde{\psi}_{\mathbf{k_2}}(t)\\ &- k_{1x}^2\tilde{\phi}_{\mathbf{k_2}}(t)\tilde{\psi}_{\mathbf{k_1}}(t) + k_{1y}^2[\tilde{\phi}_{\mathbf{k_1}}(t)\tilde{\psi}_{\mathbf{k_2}}(t)\\ &- \tilde{\phi}_{\mathbf{k_2}}(t)\tilde{\psi}_{\mathbf{k_1}}(t)]\},\end{aligned} \tag{6}$$

where $\boldsymbol{\Psi}_k=\tilde{\psi}(k_x,k_y=0)$. The small scale viscous effects here cannot dissipate the long wavelength flows. Hence the only mechanism responsible for damping (and growth) of flows is the nonlinear interaction term in Eq. (6) which imposes a rather stringent condition on the nonlinear flows dynamics, and is given as $\boldsymbol{\Lambda}_{\mathbf{k_1},\mathbf{k_2}}=k_2^2\tilde{\phi}_{\mathbf{k_1}}\tilde{\psi}_{\mathbf{k_2}}-k_1^2\tilde{\phi}_{\mathbf{k_2}}\tilde{\psi}_{\mathbf{k_1}}$. The nonlinear flow mode will therefore grow when the spectrally averaged $\boldsymbol{\Lambda}_{\mathbf{k_1},\mathbf{k_2}}$ is finite, and this is shown in Fig 5. For longer wavelengths, the nonlinear interaction parameter ($\boldsymbol{\Lambda}_{\mathbf{k_1},\mathbf{k_2}}$) becomes negligibly small and results in a weakening of the nonlinear interactions, which then leads to nonlinear damping of the flows [Figs 4 and 5].

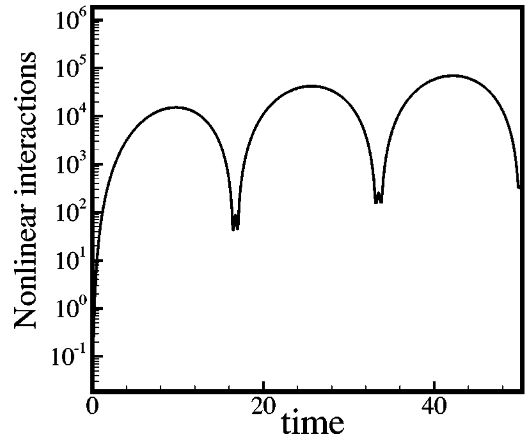

FIG. 5. Spectrally averaged nonlinear interaction term $\boldsymbol{\Lambda}_{\mathbf{k_1},\mathbf{k_2}}$ (in arbitrary units).

V. CONCLUSION

Flow generation in an Eulerian fluid has been reported [15] in a system with no unstable modes, resulting instead from infinitesimally small sheared-flow perturbations which enhance the Reynolds stresses. By contrast, we find the remarkable result that the response and interaction of acoustic modes in a fluid to and with incompressible turbulence leads to the generation of periodic nonlinear flows, driven by effective Reynolds stresses. By virtue of their structure, the NI hydro equations are ideally suited to the investigation of wave phenomena in a fully turbulent medium and our simulations here reveal the rich nonlinear complexity of this problem. The anisotropic flow generated due to nonlinear instability in our simulation convects passively the weakly compressive density fluctuations and may hint that anisotropic density fluctuations observed in the solar wind [16] in the high plasma-β regime ($\beta\simeq 1$) could possibly be explained on the basis of nearly incompressible theory [17].

ACKNOWLEDGMENTS

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